From Super Poincaré to Weighted Log-Sobolev and Entropy-Cost Inequalities *

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Abstract

We derive weighted log-Sobolev inequalities from a class of super Poincaré inequalities. As an application, the Talagrand inequality with larger distances are obtained. In particular, on a complete connected Riemannian manifold, we prove that the $\log^\delta$-Sobolev inequality with $\delta \in (1,2)$ implies the $L^{2/(2-\delta)}$-transportation cost inequality

$$W^\rho_{2/(2-\delta)}(f\mu,\mu)^{2/(2-\delta)} \le C\mu(f\log f), \quad \mu(f)=1, f\ge 0$$

for some constant $C>0$, and they are equivalent if the curvature of the corresponding generator is bounded below. Weighted log-Sobolev and entropy-cost inequalities are also derived for a large class of probability measures on $\mathbb{R}^d$.

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1 Introduction

Let (E,ρ) be a Polish space and μ a probability measure on E. For $p\ge 1$ we define the L^p-Wasserstein distance (or the L^p-transportation cost) by

$$W^\rho_p(\mu_1,\mu_2) := \left\{\inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)} \int_{E\times E} \rho(x,y)^p \pi(\mathrm{d}x,\mathrm{d}y)\right\}^{1/p}$$

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for probability measures μ_1, μ_2 on E, where $\mathscr{C}(\mu_1, \mu_2)$ is the class of probability measures on $E \times E$ with marginal distributions μ_1 and μ_2.

According to [4, Corollary 4],

$$W_p^\rho(f\mu, \mu)^{2p} \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1$$

holds for some $C > 0$ provided $\mu(e^{\lambda \rho(o,\cdot)^{2p}}) < \infty$ for some $\lambda > 0$, where $o \in E$ is a fixed point. See also [8] for $p = 1$. Furthermore, it is easy to derive from [14, Theorem 1.15] that for any $q \in [1, 2p)$, there exists $C > 0$ such that

$$W_q^\rho(f\mu, \mu)^{2p} \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1 \tag{1.1}$$

if and only if $\mu(e^{\lambda \rho(o,\cdot)^{2p}}) < \infty$ for some $\lambda > 0$. In general, however, this concentration of μ does not imply (1.1) for $q = 2p$. Indeed, there exist a plentiful examples where $\mu(e^{\lambda \rho(o,\cdot)^2}) < \infty$ for some $\lambda > 0$ but there is no any constant $C > 0$ such that the Talagrand inequality

$$W_2^\rho(f\mu, \mu)^2 \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1 \tag{1.2}$$

holds, see e.g. [1] for examples with $\mu(e^{\lambda \rho(o,\cdot)^2}) < \infty$ for some $\lambda > 0$ but the Poincaré inequality does not hold, which is weaker than (1.2) (see [17, Section 7] or [2, Section 4.1]).

Therefore, to derive (1.1) with $q = 2p$, one needs something stronger than the corresponding concentration of μ. In fact, it is now well known in the literature that, the Talagrand inequality follows from the log-Sobolev inequality for a class of local Dirichlet forms, see [21, 17, 2, 25, 20] and references within.

In this paper, we aim to derive (1.1) with $q = 2p$, i.e.

$$W_{2p}^\rho(f\mu, \mu)^{2p} \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1, \tag{1.3}$$

by using functional inequalities stronger than the log-Sobolev one.

To this end, in Section 2 we study the weighted log-Sobolev inequality

$$\mu(f^2 \log f^2) \le C\mu(\alpha \circ \rho(o,\cdot)\Gamma(f,f)), \quad \mu(f^2) = 1$$

for a positive function $\alpha(r) \to 0$ as $r \to \infty$ and a nice square field Γ. Combining this with known results on log-Sobolev and the Talagrand inequality, we derive (1.2) with the original distance ρ replaced by a larger one, which is induced by the weighted square field $\alpha \circ \rho(o,\cdot)\Gamma$. In particular, we have the following result on a Riemannian manifold.

Let M be a connected complete Riemannian manifold, and $\mu(dx) = e^{V(x)}dx$ a probability measure on M for some $V \in C(M)$. We shall use the following super Poincaré inequality (see [23])

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0 \tag{1.4}$$

to establish the corresponding weighted log-Sobolev inequality

$$\mu(f^2 \log f^2) \le C\mu(\alpha \circ \rho(o,\cdot)|\nabla f|^2), \quad \mu(f^2) = 1. \tag{1.5}$$

By [25, Theorem 1.1], (1.5) implies

$$W_2^{\rho_\alpha}(f\mu,\mu)^2 \le C\mu(f\log f), \quad f \ge 0, \mu(f^2) = 1, \tag{1.6}$$

where ρ_α is the Riemannian distance induced by the metric

$$\langle X, Y\rangle' := \frac{1}{\alpha \circ \rho(o,x)}\langle X, Y\rangle, \quad X, Y \in T_xM, \ x \in M. \tag{1.7}$$

The main result of the paper is the following.

Theorem 1.1. *Assume that (1.4) holds for some positive decreasing $\beta \in C((0,\infty))$ such that*

$$\eta(s) := \big(\log(2s)\big)\big(1 \wedge \beta^{-1}(s/2)\big), \quad s \ge 1$$

is bounded, where $\beta^{-1}(s) := \inf\{t \ge 0 : \ \beta(t) \le s\}$. Then (1.5) holds for some $C > 0$ and

$$\alpha(s) := \sup_{t \ge \mu(\rho(o,\cdot)\ge s-2)^{-1}} \eta(t), \quad s \ge 0.$$

Consequently, (1.6) holds.

The following consequences show that the above result is sharp in specific situations.

Corollary 1.2. *Let $\delta \in (1,2)$.*

(a) (1.4) with $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ implies (1.5) with

$$\alpha(s) := (1 + \rho(o,\cdot))^{-2(\delta-1)/(2-\delta)}$$

and (1.6) with $\rho_\alpha(x,y)$ replaced by

$$\rho(x,y)(1 + \rho(o,x) \vee \rho(o,y))^{(\delta-1)/(2-\delta)}.$$

Consequently, it implies

$$W^\rho_{2/(2-\delta)}(f\mu,\mu)^{2/(2-\delta)} \le C\mu(f\log f), \quad \mu(f) = 1, f \ge 0 \tag{1.8}$$

for some constant $C > 0$.

(b) If $V \in C^2(M)$ with $\mathrm{Ric} - \mathrm{Hess}_V$ *bounded below, then the following are equivalent to each other:*

(1) *(1.4) with* $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ *for some constant* $c > 0$;

(2) *(1.5) with* $\alpha(s) := (1 + \rho(o, \cdot))^{-2(\delta-1)/(2-\delta)}$ *for some* $C > 0$;

(3) *(1.6) for some* $C > 0$ *and* $\rho_\alpha(x, y)$ *replaced by* $\rho(x, y)(1+\rho(o, x) \vee \rho(o, y))^{(\delta-1)/(2-\delta)}$;

(4) *(1.8) for some* $C > 0$;

(5) $\mu(\exp[\lambda\rho(o, \cdot)^{2/(2-\delta)}]) < \infty$ *for some* $\lambda > 0$.

We remark that (1.4) with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $c > 0$ is equivalent to the following $\log^\delta$-Sobolev inequality mentioned in the abstract (see [23, 24, 13, 26] for more general results on (1.4) and the F-Sobolev inequality)

$$\mu(f^2 \log^\delta(1 + f^2)) \le C_1\mu(|\nabla f|^2) + C_2, \quad \mu(f^2) = 1.$$

Since due to [24, Corollary 5.3] if (1.4) holds with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $\delta > 2$ then M has to be compact, as a complement to Corollary 1.2 we consider the critical case $\delta = 2$ in the next Corollary.

Corollary 1.3. *(1.4) with* $\beta(r) = \exp[c(1 + r^{-1/2})]$ *for some* $c > 0$ *implies (1.5) with* $\alpha(s) := \mathrm{e}^{-c_1 s}$ *for some* $c_1 > 0$ *and (1.6) with* $\rho_\alpha(x, y)$ *replaced by*

$$\rho(x, y)\mathrm{e}^{c_2[\rho(o,x)\vee\rho(o,y)]} \ge \mathrm{e}^{c_3\rho(x,y)} - 1$$

for some $c_2, c_3 > 0$. *If* $\mathrm{Ric} - \mathrm{Hess}_V$ *is bounded below, they are all equivalent to the concentration* $\mu(\exp[\mathrm{e}^{\lambda\rho(o,\cdot)}]) < \infty$ *for some* $\lambda > 0$.

Example 1.1. Let Ric be bounded below. Let $V \in C(M)$ be such that $V + a\rho(o, \cdot)^\theta$ is bounded for some $a > 0$ and $\theta \ge 2$. By [23, Corollaries 2.5 and 3.3], (1.4) holds for $\delta = 2(\theta - 1)/\theta$. Then Corollary 1.2 implies

$$W_\theta^\rho(f\mu, \mu)^\theta \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1$$

for some constant $C > 0$.

In this inequality θ could not be replaced by any larger number, since $W_\theta^\rho \ge W_1^\rho$ and by Proposition 3.1 below for any $p \ge 1$ the inequality

$$W_1^\rho(f\mu, \mu)^p \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1$$

implies $\mu(\mathrm{e}^{\lambda\rho(o,\cdot)^p}) < \infty$ for some $\lambda > 0$, which fails when $p > \theta$ for μ specified above.

Example 1.2. In the situation of Example 1.1 but let $V + \exp[\sigma\rho(o,\cdot)]$ be bounded for some $\sigma > 0$. Then by [23, Corollaries 2.5 and 3.3], (1.4) holds with $\beta(r) = \exp[c(1+r^{-1/2})]$ for some $c > 0$. Hence, by Corollary 1.3,

$$\inf_{\pi\in\mathscr{C}(\mu,f\mu)} \int_{M\times M} \rho(x,y)^2 \mathrm{e}^{c_1\rho(x,y)} \pi(\mathrm{d}x,\mathrm{d}y) \le C\mu(f\log f), \quad f\ge 0, \mu(f) = 1 \tag{1.9}$$

holds for some $c_1, C > 0$.

On the other hand, it is easy to see from Jensen's inequality that the left hand side is larger than

$$(\exp[c_2 W_1^\rho(\mu, f\mu)] - 1)^2$$

for some $c_2 > 0$. So, by Proposition 3.1 below (1.9) implies $\mu(\exp[\exp(\lambda\rho(o,\cdot))]) < \infty$ holds for any $\lambda > 0$, which is the exact concentration property of the given measure μ.

In the next section we study the super Poincaré and the weighted log-Sobolev inequality in an abstract framework, and complete proofs of the above results are presented in Section 3. Finally, weighted log-Sobolev and transportation cost inequalities are also studied for probability measures on $\mathbb{R}^d$ by using concentrations.

2 From super Poincaré to weighted log-Sobolev inequalities

We shall work with a diffusion framework as in [1]. Let $(E, \mathscr{F}, \mu)$ be a separable complete probability space, and let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a conservative symmetric local Dirichlet form on $L^2(\mu)$ with domain $\mathscr{D}(\mathscr{E})$ in the following sense. Let $\mathscr{A}$ be a dense subspace of $\mathscr{D}(\mathscr{E})$ under the $\mathscr{E}_1^{1/2}$-norm ($\mathscr{E}_1(f,f) = \|f\|_2^2 + \mathscr{E}(f,f)$) which is composed of bounded functions, stable under products and composition with Lipschitz functions on $\mathbb{R}$. Let $\Gamma : \mathscr{A}\times\mathscr{A} \to \mathscr{M}_b$ be a bilinear mapping, where $\mathscr{M}_b$ is the set of all bounded measurable functions on E, such that

(1) $\Gamma(f,f) \ge 0$ and $\mathscr{E}(f,g) = \mu(\Gamma(f,g))$ for $f,g\in\mathscr{A}$;
(2) $\Gamma(\phi\circ f, g) = \phi'(f)\Gamma(f,g)$ for $f,g\in\mathscr{A}$ and $\phi\in C_b^\infty(\mathbb{R})$;
(3) $\Gamma(fg,h) = g\Gamma(f,h) + f\Gamma(g,h)$ for $f,g,h\in\mathscr{A}$ with $fg\in\mathscr{A}$.

It is easy to see that the positivity and the bilinear property imply $\Gamma(f,g)^2 \le \Gamma(f,f)\Gamma(g,g)$ for all $f,g\in\mathscr{A}$. For simplicity we set below $\Gamma(f,f) = \Gamma(f)$ and $\mathscr{E}(f,f) = \mathscr{E}(f)$.

We shall denote by $\mathscr{A}_{\mathrm{loc}}$ the set of functions f such that for any integer n, the truncated function $f_n = \min(n, \max(f, -n))$ is in $\mathscr{A}$. For such functions, the bilinear map Γ automatically extends and shares the same properties than for functions in $\mathscr{A}$.

Next, let $\varrho\in\mathscr{A}_{\mathrm{loc}}$ be positive such that $\Gamma(\varrho,\varrho)\le 1$. We shall start from the super Poincaré inequality

$$\mu(f^2) \le r\mathscr{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > 0. \tag{2.1}$$

To derive the desired weighted log-Sobolev inequality

$$\mu(f^2 \log f^2) \le C\mu(\Gamma(f,f)\alpha \circ \varrho), \quad \mu(f^2) = 1, \tag{2.2}$$

we shall also need the following Poincaré inequality

$$\mu(f^2) \le C_0 \mathscr{E}(f,f) + \mu(f)^2 \tag{2.3}$$

for some $C_0 > 0$. Here and in what follows, the reference function f is taken from $\mathscr{A}$.

Theorem 2.1. *Assume (2.3) holds for some $C_0 > 0$. Then (2.1) implies (2.2) for some constant $C > 0$ and α given in Theorem 1.1.*

Proof. (a) Let $\Phi(s) = \mu(\varrho \ge s)$ which decreases to zero as $s \to \infty$. We may take $r_0 > 0$ such that

$$r_0(1 + \sup_{s \ge 1} \eta(s)) \le \frac{1}{32} \tag{2.4}$$

and

$$\beta^{-1}(e^{r_0^{-1}}/4) \le 1. \tag{2.5}$$

For a fixed number $r \in (0, r_0]$ we define

$$\begin{aligned} h_n &= \big((\varrho - \Phi^{-1}(2e^{-r^{-1}}) - n)_+ \wedge 1\big)\big((n + 2 + \Phi^{-1}(2e^{-r^{-1}}) - \varrho)_+ \wedge 1\big), \\ \delta_n &= \Big(\log \frac{2}{\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))}\Big)\beta^{-1}\Big(\frac{1}{2\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))}\Big), \\ B_n &= \{n \le \varrho - \Phi^{-1}(2e^{-r^{-1}}) \le n + 2\}, \qquad n \ge 0. \end{aligned}$$

Then

$$\sum_{n=0}^{\infty} h_n^2 \ge \frac{1}{2} 1_{\{\rho \ge 1 + \Phi^{-1}(2e^{-r^{-1}})\}}. \tag{2.6}$$

By (2.1) and noting that

$$\mu(|f|h_n)^2 \le \mu(f^2h_n^2)\mu(\varrho > n + \Phi^{-1}(2\mathrm{e}^{-r^{-1}})) \le \mu(f^2h_n^2)\Phi(n + \Phi^{-1}(2\mathrm{e}^{-r^{-1}})),$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mu(f^2h_n^2) &\le \sum_{n=0}^{\infty}\Big\{r_n\mu\big(\Gamma(fh_n, fh_n)\big) + \beta(r_n)\mu(|f|h_n)^2\Big\}\\ &\le \sum_{n=0}^{\infty}\Big\{\frac{2r_n}{\delta_n}\mu(\Gamma(f,f)\delta_n 1_{B_n}) + 2r_n\mu(f^2 1_{B_n}) + \beta(r_n)\Phi(n+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))\mu(f^2h_n^2)\Big\}\end{aligned}$$

for $r_n > 0$. Since by (2.5) and the definition of α

$$\alpha(s) \ge \delta_n \text{ for } s \ge n + 2 + \Phi^{-1}(2\mathrm{e}^{-r^{-1}}),$$

letting $r_n = \delta_n r$ we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mu(f^2h_n^2) \le \sum_{n=0}^{\infty}\Big\{&2r\mu(\Gamma(f,f)\alpha\circ\varrho 1_{B_n}) + 2r\delta_n\mu(f^2 1_{B_n})\\ &+ \beta(r\delta_n)\Phi(n+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))\mu(f^2h_n^2)\Big\}.\end{aligned} \tag{2.7}$$

Noting that

$$A := r\log\frac{2}{\Phi(n+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))} \ge r\log\frac{2}{\Phi(\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))} = 1,$$

we have

$$\beta(\delta_n r) = \beta\Big(A\beta^{-1}\Big(\frac{1}{2\Phi(n+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))}\Big)\Big) \le \frac{1}{2\Phi(n+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))}.$$

Thus, by (2.7) and (2.4) and the fact that $\delta_n \le \sup\eta$, we arrive at

$$\sum_{n=0}^{\infty}\mu(f^2h_n^2) \le \sum_{n=0}^{\infty}\Big\{2r\mu(\Gamma(f,f)\alpha\circ\varrho 1_{B_n}) + \frac{1}{8}\mu(f^2) + \frac{1}{2}\sum_{n=0}^{\infty}\mu(f^2h_n^2).$$

It follows from this and (2.6) that

$$\mu(f^2 1_{\{\varrho\ge 1+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\}}) \le 8r\mu(\Gamma(f,f)\alpha\circ\varrho) + \frac{1}{2}\mu(f^2). \tag{2.8}$$

(b) On the other hand, since α is decreasing

$$\begin{aligned}&\mu(f^2 1_{\{\varrho\le 1+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\}}) \le \mu(f^2\{(2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})-\varrho)_+^2\wedge 1\})\\
&\le 2s\mu(\Gamma(f,f)1_{\{\varrho\le 2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\}}) + 2s\mu(f^2)+\beta(s)\mu(|f|)^2\\
&\le \frac{2s}{\alpha(2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))}\mu(\Gamma(f,f)\alpha\circ\varrho)+2s\mu(f^2)+\beta(s)\mu(|f|)^2,\quad s>0.\end{aligned}$$

Taking

$$s=r\alpha(2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))\le\frac{1}{32}$$

due to (2.4), we obtain

$$\mu(f^2 1_{\{\varrho\le 1+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\}}) \le 2r\mu(\Gamma(f,f)\alpha\circ\varrho)+\frac{1}{16}\mu(f^2)+\beta\big(r\alpha(2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))\big)\mu(|f|)^2.$$

Since by (2.5) and the definition of α

$$\begin{aligned}r\alpha\big(2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\big) &\ge \Big(r\log\frac{2}{\Phi(\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))}\Big)\beta^{-1}\Big(\frac{1}{2\Phi(\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))}\Big)\\
&=\beta^{-1}\Big(\frac{\mathrm{e}^{r^{-1}}}{4}\Big),\end{aligned}$$

we obtain

$$\mu(f^2 1_{\{\varrho\le 1+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\}}) \le 2r\mu(\Gamma(f,f)\alpha\circ\varrho)+\frac{1}{16}\mu(f^2)+\frac{\mathrm{e}^{r^{-1}}}{4}\mu(|f|)^2.$$

Combining this with (2.8) we conclude that

$$\mu(f^2)\le 40r\mu(\Gamma(f,f)\alpha\circ\varrho)+\mathrm{e}^{r^{-1}}\mu(|f|)^2,\quad r\in(0,r_0].$$

Therefore, there exists a constant $c>0$ such that

$$\mu(f^2)\le r\mu(\Gamma(f,f)\alpha\circ\varrho)+\mathrm{e}^{c(1+r^{-1})}\mu(|f|)^2,\quad r>0. \tag{2.9}$$

According to e.g. [24, Corollary 1.3], this is equivalent to the defective weighted log-Sobolev inequality

$$\mu(f^2\log f^2)\le C_1\mu(\Gamma(f,f)\alpha\circ\varrho)+C_2,\quad \mu(f^2)=1. \tag{2.10}$$

(c) Finally, for any f with $\mu(f)=0$, it follows from (2.3) that

$$
\begin{aligned}
\mu(f^2) &\le \mu(f^2\{(1+R-\varrho)_+^2\wedge 1\}) + \|f\|_\infty^2\mu(\varrho\ge R)\\
&\le 2C_0\mu(\Gamma(f,f)1_{\{\varrho\le 1+R\}}) + (2C_0+1)\|f\|_\infty^2\mu(\varrho\ge R) + \mu(f\{(\varrho-R)_+\wedge 1\})^2\\
&\le \frac{2C_0}{\alpha(1+R)}\mu(\Gamma(f,f)\alpha\circ\varrho) + 2(C_0+1)\|f\|_\infty^2\mu(\varrho\ge R), \quad R>0.
\end{aligned}
$$

Since $\mu(\varrho\ge R)\to 0$ as $R\to\infty$, the weighted weak Poincaré inequality

$$\mu(f^2)\le\tilde\beta(r)\mu(\Gamma(f,f)\alpha\circ\varrho) + r\|f\|_\infty^2, \quad r>0, \mu(f)=0$$

holds for some positive function $\tilde\beta$ on $(0,\infty)$. By [19, Propsosition 1.3], this and (2.9) implies the weighted Poincaré inequality

$$\mu(f^2)\le C'\mu(\Gamma(f,f)\alpha\circ\varrho) + \mu(f)^2$$

for some constant $C'>0$. Combining this with (2.10) we obtain the desired weighted log-Sobolev inequality (2.2). □

3 Proofs of Theorem 1.1 and Corollaries

Proof of Theorem 1.1. Since α is bounded, the completeness of the original metric implies that of the weighted one given by (1.7). So, (1.6) follows from (1.5) due to [25, Theorem 1.1] with $p\to 2$. Thus, by Theorem 2.1 with $E=M$ and $\Gamma(f,f)=|\nabla f|^2$, it suffices to prove that (1.4) implies the Poincaré inequality (2.3) for some $C_0>0$. Due to [23] the super Poincaré inequality (1.4) implies that the spectrum of L is discrete. Moreover, since M is connected, the corresponding Dirichlet form is irreducible so that 0 is a simple eigenvalue. Therefore, L possesses a spectral gap, which is equivalent to the desired Poincaré inequality. □

To complete the proof of Corollary 1.2, in the spirit of [16, 3] we introduce below a deviation inequality induced by the L^1-transportation cost inequality.

Proposition 3.1. *Let $\tilde\rho: M\times M\to[0,\infty)$ be measurable. For any $r>0$ and measurable set $A\subset M$ with $\mu(A)>0$, let*

$$A_r=\{x\in M:\tilde\rho(x,y)\ge r \textit{ for some } y\in A\},\quad r>0.$$

If

$$W_1^{\tilde\rho}(f\mu,\mu)\le\Phi\circ\mu(f\log f),\quad f\ge 0,\mu(f)=1 \tag{3.1}$$

holds for some positive increasing $\Phi\in C([0,\infty))$, then

$$\mu(A_r) \le \exp\big[-\Phi^{-1}(r - \Phi\circ\log\mu(A)^{-1})\big], \quad r > \Phi\circ\log\mu(A)^{-1}, \tag{3.2}$$

where $\Phi^{-1}(r) := \inf\{s\ge 0:\ \Phi(s)\ge r\},\ r\ge 0.$

Proof. It suffices to prove for $\mu(A_r)>0$. In this case, letting $\mu_A=\mu(\cdot\cap A)/\mu(A)$ and $\mu_{A_r}=\mu(\cdot\cap A_r)/\mu(A_r)$, we obtain from (3.1) that

$$r\le W_1^{\tilde\rho}(\mu_A,\mu_{A_r})\le W_1^{\tilde\rho}(\mu_A,\mu)+W_1^{\tilde\rho}(\mu_{A_r},\mu)\le \Phi\circ\log\mu(A)^{-1}+\Phi\circ\log\mu(A_r)^{-1}.$$

This completes the proof. □

Proof of Corollary 1.2. (a) Let $\beta(r)=\mathrm{e}^{c(1+r^{-1/\delta})}$ for some $c>0$ and $\delta>1$. It is easy to see that

$$1\wedge\beta^{-1}(s/2)\le c_1\log^{-\delta}(2s),\quad s\ge 1$$

holds for some constant $c_1>0$. Next, by [24, Corollary 5.3], (1.4) with this specific function β implies

$$\mu(\rho(o,\cdot)\ge s-2)\le c_2\exp[-c_3 s^{2/(2-\delta)}],\quad s\ge 0$$

for some constants $c_2,c_3>0$. Therefore,

$$\alpha(s)\le c_4(1+s)^{-2(\delta-1)/(2-\delta)},\quad s\ge 0 \tag{3.3}$$

holds for some constant $c_4>0$.

On the other hand, for any $x_1,x_2\in M$ let $i\in\{1,2\}$ such that $\rho(o,x_i)=\rho(o,x_1)\vee\rho(o,x_2)$. Define

$$f(x)=\Big(\rho(x,x_i)\wedge\frac{\rho(o,x_i)}{2}\Big)(1+\rho(o,x_i))^{(\delta-1)/(2-\delta)},\quad x\in\mathbb{R}^d.$$

Then

$$\begin{aligned}&\alpha\circ\rho(o,\cdot)|\nabla f|^2\le c_4(1+\rho(o,\cdot))^{-2(\delta-1)/(2-\delta)}|\nabla f|^2\\ &\le c_4 1_{\{\rho(o,x_i)/2\le\rho(o,\cdot)\le 3\rho(o,x_i)/2\}}(1+\rho(o,\cdot))^{-2(\delta-1)/(2-\delta)}(1+\rho(o,x_i))^{2(\delta-1)/(2-\delta)}\le c_5\end{aligned}$$

for some constant $c_5>0$. Since by the triangle inequality $\rho(o,x_i)\ge\frac12\rho(x_1,x_2)$, this implies that the intrinsic distance ρ_α satisfies

$$\begin{aligned}&\rho_\alpha(x_1,x_2)^2\ge\frac{|f(x_1)-f(x_2)|^2}{c_5}\\ &\ge c_6\rho(x_1,x_2)^2(1+\rho(o,x_1)\vee\rho(o,x_2))^{2(\delta-1)/(2-\delta)}\ge c_7\rho(x_1,x_2)^{2/(2-\delta)}\end{aligned}$$

for some constant $c_6, c_7 > 0$. Hence the proof of (a) is completed by Theorem 1.1.

(b) Now, assume that

$$\mathrm{Ric} - \mathrm{Hess}_V \geq -K$$

for some $K \geq 0$. By (a) and Proposition 3.1, which ensures the implication from (4) to (5), it suffices to deduce (1) from (5). Let

$$h(r) = \mu(\mathrm{e}^{r\rho(o,\cdot)^2}), \qquad r > 0.$$

By [24, Theorem 5.7], the super Poincaré inequality (1.4) holds with

$$\beta(r) := c_0 \inf_{0<r_1<r} r_1 \inf_{s>0} \frac{1}{s} h(2K + 12s^{-1})\mathrm{e}^{s/r_1 - 1}, \quad r > 0 \tag{3.4}$$

for some constant $c_0 > 0$. Since for any $\lambda > 0$ there exists $c(\lambda) > 0$ such that

$$rt^2 \leq \lambda t^{2/(2-\delta)} + c(\lambda) r^{1/(\delta-1)}, \quad r > 0,$$

it follows from (5) that

$$h(r) \leq c_1 \exp[c_1 r^{1/(\delta-1)}], \quad r > 0$$

for some constants $c_1 > 0$. Therefore,

$$\beta(r) \leq c_2 \inf_{0<r_1<r} r_1 \inf_{s>0} \frac{1}{s} \exp[c_2 s^{-1/(\delta-1)} + s/r_1], \quad r > 0$$

for some $c_2 > 0$. Taking $s = r^{(\delta-1)/\delta}$ and $r_1 = r$, we conclude that

$$\beta(r) \leq \mathrm{e}^{c(1+r^{-1/\delta})}, \quad r > 0$$

for some $c > 0$. Thus, (1) holds. $\square$

Proof of Corollary 1.3. The proof is similar to that of Corollary 1.2 by noting that (1.4) with $\beta(r) = \exp[c(1 + r^{-1/2})]$ implies $\mu(\rho(o, \cdot) \geq s) \leq \exp[-c\mathrm{e}^{c_1 s}]$ for some $c_1 > 0$, see [24, Corollary 5.3]. $\square$

4 Weighted log-Sobolev and transportation cost inequalities on $\mathbb{R}^d$

Our main purpose of this section is to establish the weighted log-Sobolev inequality for an arbitrary probability measure using the concentration of this measure. We shall also prove the HWI inequality introduced in [2] for the corresponding weighted Dirichlet form. The main point is to find square fields (resp. cost functions) for a given probability measure to satisfy the log-Sobolev inequality (resp. the Talagrand transportation cost inequality).

So, the line of our study is exactly opposed to existed references in the literature, see e.g. [9, 10, 11] and references within, which provided conditions on the reference measure such that the log-Sobolev (resp. transportation cost) inequality holds for a given square field (resp. the corresponding cost function).

The basic idea of the study comes from Caffarelli [5] which says that for any probability measure $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ on $\mathbb{R}^d$, there exists a convex function ψ on $\mathbb{R}^d$ such that $\nabla\psi$ pushes μ forward to the standard Gaussian measure γ; that is, letting

$$y(x) := \nabla\psi(x), \quad x \in \mathbb{R}^d,$$

which is one-to-one, one has $\gamma = \mu \circ y^{-1}$. Furthermore, $\nabla\psi$ is uniquely determined and Hess_ψ is non-degenerate with

$$\det(\mathrm{Hess}_\psi) = (2\pi)^{d/2}\mathrm{e}^{V+|\nabla\psi|^2/2}.$$

Let

$$\rho(x_1, x_2) := |y(x_1) - y(x_2)|, \quad x_1, x_2 \in \mathbb{R}^d.$$

Let W_2 be the L^2-Wasserstein distance induced by the usual Euclidiean metric. Due to Talagrand [21]

$$W_2(\gamma, f^2\gamma)^2 \le 2\gamma(f^2 \log f^2), \quad \gamma(f^2) = 1. \tag{4.1}$$

Since $\pi \in \mathscr{C}(\mu \circ y^{-1}, (f^2 \circ y^{-1})\mu \circ y^{-1})$ if and only if $\pi \circ (y \otimes y) \in \mathscr{C}(\mu, f^2\mu)$, we obtain from (4.1) and the change of variables theorem that

$$W_2^\rho(\mu, f^2\mu)^2 = W_2(\gamma, (f^2 \circ y^{-1})\gamma)^2 \le 2\gamma(f^2 \circ y^{-1} \log f^2 \circ y^{-1}) = 2\mu(f^2 \log f^2), \quad \mu(f^2) = 1.$$

Similarly, since

$$\nabla(f \circ y^{-1}) = (Dy^{-1})(\nabla f) \circ y^{-1} = [(Dy) \circ y^{-1}]^{-1}(\nabla f) \circ y^{-1} = [(\mathrm{Hess}_\psi)^{-1}\nabla f] \circ y^{-1},$$

where $Dy := (\partial_i y_j)_{d\times d}$, by Gross' log-Sobolev inequality for γ (see [12]) we obtain

$$\mu(f^2 \log f^2) \le 2\mu(|(\mathrm{Hess}_\psi)^{-1}\nabla f|^2), \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1.$$

On the other hand, however, since the transportation $\nabla\psi$ is normally inexplicit, it is hard to estimate the distance ρ and the matrix Hess_ψ. So, to derive transportation and log-Sobolev inequalities with explicit distances and Dirichlet forms, we shall construct, instead of $\nabla\psi$, an explicit map using the concentration of μ, which transports the measure into the standard Gaussian measure with a perturbation. In many cases this perturbation is bounded and hence, does not make much trouble to derive the desired inequalities.

4.1 Main results

In this subsection we provide an explicit positive function α and an explicit distance ρ on $\mathbb{R}^d$ such that the log-Sobolev inequality

$$(4.2) \qquad \mu(f^2 \log f^2) \le 2\mu(\alpha|\nabla f|^2), \quad f \in C_0^\infty(\mathbb{R}^d),\ \mu(f^2) = 1$$

and the transportation-cost inequality

$$(4.3) \qquad W_2^\rho(\mu, f^2\mu)^2 \le 2\mu(f^2 \log f^2), \quad \mu(f^2) = 1$$

hold. In a special case, we are also able to present the HWI inequality stronger than (4.2).

Let us first consider a probability measure $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ on $[\delta, \infty)$ for some $\delta \in [-\infty, \infty)$, where $[-\infty, \infty)$ is regarded as $\mathbb{R}$. Let

$$\Phi_\delta(r) := \frac{1}{c_\delta}\int_\delta^r \mathrm{e}^{-s^2/2}\mathrm{d}s, \quad \varphi(r) := \mu([\delta, r)) = \int_\delta^r \mathrm{e}^{V(x)}\mathrm{d}x, \quad r \ge \delta,$$

where $c_\delta := \int_\delta^\infty \mathrm{e}^{-x^2/2}\mathrm{d}x$ is the normalization.

Theorem 4.1. *Let $\mu(\mathrm{d}x) := 1_{[\delta,\infty)}(x)\mathrm{e}^{V(x)}\mathrm{d}x$ be a probability measure on $[\delta, \infty)$. For the above defined Φ_δ and φ, (4.2) and (4.3) hold with $\mathbb{R}^d$ replaced by $[\delta, \infty)$ for*

$$\begin{aligned} \alpha &:= \Big(\frac{\Phi_\delta' \circ \Phi_\delta^{-1} \circ \varphi}{\varphi'}\Big)^2, \\ \rho(x, y) &:= |\Phi_\delta^{-1} \circ \varphi(x) - \Phi_\delta^{-1} \circ \varphi(y)|, \quad x, y \ge \delta. \end{aligned}$$

Furthermore,

$$(4.4) \quad \mu(f^2 \log f^2) + W_2^\rho(\mu, f^2\mu)^2 \le 2\sqrt{2\mu(\alpha f'^2)}\,W_2^\rho(\mu, f^2\mu), \quad f \in C_0^\infty([\delta, \infty)), \mu(f^2) = 1.$$

The inequality (4.4), linking the Wasserstein distance, the relative entropy and the energy, is called the HWI inequality in [2] and [18].

To extend this result to $\mathbb{R}^d$ for $d \ge 2$, we consider the polar coordinate $(r, \theta) \in [0, \infty) \times \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ with the induced metric. Then μ can be represented as

$$\mathrm{d}\mu = c(d)r^{d-1}\mathrm{e}^{V(r\theta)}\mathrm{d}r\mathrm{d}\theta =: G(r, \theta)\mathrm{d}r\mathrm{d}\theta,$$

where $\mathrm{d}\theta$ is the normalized volume measure on $\mathbb{S}^{d-1}$, and $c(d)/d$ equals to the volume of the unit ball in $\mathbb{R}^d$. Let $B(0, r) := \{x \in \mathbb{R}^d : |x| < r\}$ and

$$
\begin{aligned}
\Phi_0(r) &:= \int_{B(0,r)} \frac{\mathrm{e}^{-|x|^2/2}\mathrm{d}x}{(2\pi)^{d/2}}, \quad r \geq 0,\\
h(\theta) &:= \int_0^\infty s^{d-1}\mathrm{e}^{V(s\theta)}\mathrm{d}s, \quad \theta \in \mathbb{S}^{d-1},\\
\varphi_\theta(r) &:= \frac{1}{h(\theta)}\int_0^r s^{d-1}\mathrm{e}^{V(s\theta)}\mathrm{d}s, \quad \theta \in \mathbb{S}^{d-1}, r \geq 0.
\end{aligned}
$$

Since $\mu(\mathbb{R}^d) = 1$, we have $h(\theta) \in (0, \infty)$ for a.e. $\theta \in \mathbb{S}^{d-1}$.

We shall prove that the map

$$
x \mapsto \Phi_0^{-1} \circ \varphi_{\frac{x}{|x|}}(|x|)\frac{x}{|x|}
$$

transports μ into a Gaussian measure with density $h \circ \theta$. Thus, to derive the desired inequalities for μ, we need a regularity property of this transportation specified in the following result.

Theorem 4.2. *Let $r(x) := |x|, \theta(x) := \frac{x}{|x|}, x \in \mathbb{R}^d$. If $C(h) := \sup_{\theta_1,\theta_2\in\mathbb{S}^{d-1}} \frac{h(\theta_1)}{h(\theta_2)} < \infty$, then (4.3) holds for*

$$
\rho(x_1, x_2) := C(h)^{-1/2}|(\Phi_0^{-1} \circ \varphi_\theta(r)\theta)(x_1) - (\Phi_0^{-1} \circ \varphi_\theta(r)\theta)(x_2)|, \quad x_1, x_2 \in \mathbb{R}^d.
$$

If moreover $\varphi_\theta(r)$ is differentiable in θ then (4.2) holds for

$$
\alpha := C(h) \inf_{\varepsilon>0} \max\Big\{\frac{(1+\varepsilon)r^2}{(\Phi_0^{-1} \circ \varphi_\theta(r))^2}, \ \frac{(\Phi_0' \circ \Phi_0^{-1} \circ \varphi_\theta(r))^2}{(\varphi_\theta{}'(r))^2} + \frac{(1+\varepsilon^{-1})|\nabla_\theta\varphi_\theta(r)|^2}{(\varphi_\theta{}'(r)\Phi_0^{-1} \circ \varphi_\theta(r))^2}\Big\}.
$$

If, in particular, h is constant (it is the case if $V(x)$ depends only on $|x|$), then the following HWI inequality holds:

$$
(4.5)\quad \mu(f^2 \log f^2) + W_2^\rho(\mu, f^2\mu)^2 \leq 2\sqrt{2\mu(\alpha|\nabla f|^2)}\, W_2^\rho(\mu, f^2\mu), \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1,
$$

for

$$
\alpha := \max\Big\{\frac{r^2}{(\Phi_0^{-1} \circ \varphi(r))^2}, \ \frac{(\Phi_0' \circ \Phi_0^{-1} \circ \varphi(r))^2}{(\varphi'(r))^2}\Big\}
$$

and $\varphi = \varphi_\theta$ is independent of θ.

Note that if V is locally bounded and $\zeta(r) := \sup_{|x|=r} V(x)$ satisfies $\int_0^\infty r^{d-1}\mathrm{e}^{\zeta(r)}\mathrm{d}r < \infty$, then $C(h) < \infty$. Thus, Theorem 4.2 applies to a large number of probability measures. In particular, we have the following concrete result.

Corollary 4.3. *Let V be differentiable such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\,dx$ is a probability measure and*

$$-c_1|x|^{\delta-1} \le \langle \nabla V(x), \nabla |x| \rangle \le -c_2 |x|^{\delta-1} \tag{4.6}$$

holds for some constants $\delta, c_1, c_2 > 0$ and large $|x|$. If there exists a constant $c_3 > 0$ such that

$$|\nabla_\theta V| \le c_3, \tag{4.7}$$

where ∇_θ is the gradient on $\mathbb{S}^{d-1}$ at point θ, then there exists a constant $c > 0$ such that

$$\mu(f^2 \log f^2) \le c\mu((1 + |\cdot|)^{2-\delta} |\nabla f|^2), \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1. \tag{4.8}$$

Consequently,

$$W_2^{\tilde\rho}(\mu, f^2\mu)^2 \le c'\mu(f^2 \log f^2), \qquad \mu(f^2) = 1 \tag{4.9}$$

holds for some constant $c' > 0$ and

$$\tilde\rho(x, y) := \frac{|x - y|}{(1 + |x| \vee |y|)^{1-\delta/2}}, \quad x, y \in \mathbb{R}^d.$$

Remark. (a) The inequalities presented in Corollary 4.3 are sharp in the sense that (4.9) (and hence also (4.8)) implies $\mu(\mathrm{e}^{\lambda r^\delta}) < \infty$ for some $\lambda > 0$, which is the exact concentration of μ. This follows from [3, Corollary 3.2] and the fact that $\tilde\rho(0, x) \approx |x|^{\delta/2}$ for large $|x|$.

(b) When V is strictly concave, the matrix

$$\Lambda[v_1, v_2] := \int_0^1 s(-\mathrm{Hess}_V)((1 - s)v_1 + sv_2)\mathrm{d}s$$

is strictly positive definite for any $v_1, v_2 \in \mathbb{R}^d$. It is proved by Kolesnikov (see [15, Corollary 3.1]) that

$$\mu(f^2 \log f^2) \le \int_{\mathbb{R}^d} \langle \Lambda[T_f, \cdot]^{-1} \nabla f, \nabla f \rangle \mathrm{d}\mu, \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1, \tag{4.10}$$

where $x \mapsto T_f(x)$ is the optimal transport of $f^2\mu$ to μ. In particular, for $V(x) := -|x|^\delta + c$ with $\delta > 2$ and a constant c, [15, Example 3.2] implies (4.8) for even smooth function f^2. But Corollary 4.3 works for more general V and all smooth function f.

(c) Recently, Gentil, Guillin and Miclo [9] (see [10, 11] for further study) established a Talagrand type inequality for $V(x) = -|x|^\delta + c$ with $\delta \in [1,2]$ and a constant c. Precisely, there exist constants $a, D > 0$ such that

$$\inf_{\pi \in \mathscr{C}(\mu, f^2\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} L_{a,D}(x-y)\pi(\mathrm{d}x, \mathrm{d}y) \le D\mu(f^2 \log f^2), \qquad \mu(f^2) = 1, \tag{4.11}$$

where

$$L_{a,D}(x) := \begin{cases} \frac{|x|^2}{2}, & \text{if } |x| \le a, \\ \frac{a^{2-\delta}}{\delta}|x|^\delta + \frac{a^2(\delta-2)}{2\delta}, & \text{otherwise.} \end{cases}$$

Since $L_{a,D}(x-y) \ge \varepsilon \tilde{\rho}(x,y)^2$ for some constant $\varepsilon > 0$, this inequality implies (4.9) for $\delta \in [1,2]$. But (4.11) is yet unavailable for $\delta \notin [1,2]$ while (4.9) holds for more general V. In particular, if $\delta > 2$ then (4.9) with $\tilde{\rho}(x,y) \ge c(|x-y| \vee |x-y|^{\delta/2})$ for some $c > 0$, which is new in the literature.

4.2 Proofs

We first briefly prove for the one-dimensional case (i.e. Theorem 4.1), then extend the argument to high dimensions. It turns out, comparing with the one-dimensional case, that the difficulty point of the proof for high dimensions comes from the angle part. So, a restriction concerning the angle part was made in Theorem 4.2.

Proof of Theorem 4.1. Let $y(x) := \Phi_\delta^{-1} \circ \varphi(x),\ x \ge \delta$. We have

$$\begin{aligned} \frac{\mathrm{d}\mu}{\mathrm{d}y} &= \frac{\mathrm{d}\mu}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{e}^{V(x)} \frac{\mathrm{d}\varphi^{-1} \circ \Phi_\delta(y)}{\mathrm{d}y} \\ &= \frac{\mathrm{e}^{V(x)} \Phi_\delta'(y)}{\varphi' \circ \varphi^{-1} \circ \Phi_\delta(y)} = \frac{\mathrm{e}^{V(x)} \Phi_\delta'(y)}{\varphi'(x)} = \Phi_\delta'(y). \end{aligned}$$

Therefore, μ is the standard Gaussian measure under the new coordinate $y \in [\delta, \infty)$. In other words, one has

$$\gamma(\mathrm{d}x) := (\mu \circ y^{-1})(\mathrm{d}x) = Z 1_{[\delta,\infty)}(x) \mathrm{e}^{-x^2/2} \mathrm{d}x,$$

where Z is the normalization constant. By the HWI inequality proved in [2, 17, 18] and the Gross log-Sobolev inequality which implies the Talagrand inequality, we have

$$\begin{aligned} &\gamma(g^2 \log g^2) + W_2(\gamma, g^2\gamma)^2 \le 2\sqrt{2\gamma((g')^2)}\, W_2(\gamma, g^2\gamma), \\ &W_2(\gamma, g^2\gamma)^2 \le 2\gamma(g^2 \log g^2), \quad \gamma(g^2) = 1. \end{aligned} \tag{4.12}$$

We remark that although the HWI and Gross's log-Sobolev inequalities are stated in the above references for the global Gaussian measure, they are also true on a regular convex domain Ω, since the stronger gradient estimate

$$|\nabla P_t f| \le \mathrm{e}^{-t} P_t |\nabla f|, \quad f \in C_b^1(\Omega)$$

holds for the Neumann heat semigroup on Ω (cf. [22] and references within).

For any $f \in C_0^1([\delta, \infty))$ with $\mu(f^2) = 1$, let $g := f \circ y^{-1}$. We have

$$\frac{\mathrm{d}g}{\mathrm{d}x} = (f' \circ y^{-1}) \frac{\mathrm{d}y^{-1}}{\mathrm{d}x} = \frac{f' \circ y^{-1}}{y' \circ y^{-1}} = (f' \circ y^{-1}) \Big(\frac{\Phi_\delta' \circ \Phi_\delta^{-1} \circ \varphi}{\varphi'} \Big) \circ y^{-1}.$$

Since $\gamma = \mu \circ y^{-1}$, this and (4.12) imply (4.3) and (4.4). Finally, (4.2) is implied by (4.4). $\square$

Proof of Theorem 4.2. Let (r, θ) be the polar coordinate introduced in Section 2, and let ∇_θ denote the gradient operator on $\mathbb{S}^{d-1}$ for the standard metric induced by the Euclidean metric on $\mathbb{R}^d$. By the orthogonal decomposition of the gradient, we have

$$\nabla f = (\partial_r f) \frac{\partial}{\partial r} + r^{-1} \nabla_\theta f, \quad |\nabla f|^2 = (\partial_r f)^2 + r^{-2} |\nabla_\theta f|^2. \tag{4.13}$$

Let us introduce a new polar coordinate $(\bar{r}, \theta)$, where

$$\bar{r}(r, \theta) := \Phi_0^{-1} \circ \varphi_\theta(r), \quad r \ge 0, \theta \in \mathbb{S}^{d-1}.$$

We have

$$\mathrm{d}\mu := G(r, \theta) \mathrm{d}r \mathrm{d}\theta = \frac{G(r, \theta)}{\partial_r \bar{r}} \mathrm{d}\bar{r} \mathrm{d}\theta = c(d) h(\theta) \Phi_0'(\bar{r}) \mathrm{d}\bar{r} \mathrm{d}\theta = c(d) h(\theta) \mathrm{d}\mu_0,$$

where $\mathrm{d}\mu_0 := \Phi_0'(\bar{r}) \mathrm{d}\bar{r} \mathrm{d}\theta$ is the standard Gaussian measure under the new polar coordinate $(\bar{r}, \theta)$. Thus, letting

$$y(x) := \bar{r}(x) \theta(x) = \Phi_0^{-1} \circ \varphi_{\frac{x}{|x|}}(|x|) \theta(x), \quad x \in \mathbb{R}^d,$$

we have

$$(\mu \circ y^{-1})(\mathrm{d}x) = c(d) h(x/|x|) (\mu_0 \circ y^{-1})(\mathrm{d}x) = c(d) h(x/|x|) \gamma(\mathrm{d}x),$$

where γ is the standard Gaussian measure on $\mathbb{R}^d$. By Gross' log-Sobolev inequality one has

$$\gamma(g^2 \log g^2) \le 2\gamma(|\nabla g|^2), \quad g \in C_0^\infty(\mathbb{R}^d), \mu_0(g^2) = 1.$$

Thus, by the perturbation of the log-Sobolev inequality (cf. [7]), we have

$$(4.14)\quad (\mu\circ y^{-1})(g^2\log g^2)\le 2C(h)(\mu\circ y^{-1})(|\nabla g|^2),\quad g\in W^{2,1}(\gamma),(\mu\circ y^{-1})(g^2)=1.$$

Moreover, by [2, Corollary 3.1], (4.14) implies

$$(4.15)\qquad W_2(\mu\circ y^{-1},g^2\mu\circ y^{-1})^2\le 2C(h)(\mu\circ y^{-1})(g^2\log g^2),\quad (\mu\circ y^{-1})(g^2)=1.$$

This implies (4.3) for the desired distance ρ by using the change of variables theorem as explained above.

Similarly, to prove (4.2) we intend apply (4.14) for $g:=f\circ y^{-1}$, where $f\in C_0^\infty(\mathbb{R}^d)$ with $\mu(f^2)=1$. Since $y^{-1}=(\varphi_\theta^{-1}\circ\Phi_0(r),\theta)$ under the polar coordinate, by the chain rule we have

$$\nabla_\theta(f\circ y^{-1})=\nabla_\theta f(\varphi_\theta^{-1}\circ\Phi_0(r),\theta)=\big((\nabla_\theta f)\circ y^{-1}+(\partial_r f)\circ y^{-1}\big)\nabla_\theta\varphi_\theta^{-1}\circ\Phi_0(r).$$

But $\varphi_\theta\circ\varphi_\theta^{-1}\circ\Phi_0=\Phi_0$ implies

$$(\nabla_\theta\varphi_\theta)(\varphi_\theta^{-1}\circ\Phi_0(r))+\varphi_\theta{}'\circ\varphi_\theta^{-1}\circ\Phi_0(r)\cdot\nabla_\theta(\varphi_\theta^{-1}\circ\Phi_0(r))=0,$$

where $(\nabla_\theta\varphi_\theta)(\varphi_\theta^{-1}\circ\Phi_0(r)):=\nabla_\theta\varphi_\theta(s)|_{s=\varphi_\theta^{-1}\circ\Phi_0(r)}$, we arrive at

$$(4.16)\qquad \begin{aligned}&|\nabla_\theta(f\circ y^{-1})|^2\\ &\le(1+\varepsilon)(\partial_r f)^2\circ y^{-1}\Big(\frac{|\nabla_\theta\varphi_\theta(r)|(\varphi_\theta^{-1}\circ\Phi_0(r))}{\varphi_\theta{}'\circ\varphi_\theta^{-1}\circ\Phi_0(r)}\Big)^2+(1+\varepsilon^{-1})|\nabla_\theta f|^2\circ y^{-1}\\ &=(1+\varepsilon)(\partial_r f)^2\circ y^{-1}\Big(\frac{|\nabla_\theta\varphi_\theta(r)|}{\varphi_\theta{}'(r)}\Big)^2\circ y^{-1}+(1+\varepsilon^{-1})|\nabla_\theta f|^2\circ y^{-1}\end{aligned}$$

for any $\varepsilon>0$.

On the other hand,

$$\partial_r(f\circ y^{-1})=(\partial_r f)\circ y^{-1}\frac{\Phi_0'(r)}{\varphi_\theta{}'\circ\varphi_\theta^{-1}\circ\Phi_0(r)}.$$

Since

$$(4.17)\qquad r=\Phi_0^{-1}\circ\varphi_\theta(r(y^{-1}))=\Phi_0^{-1}\circ\varphi_\theta(r)\circ y^{-1},$$

we have

$$\Phi_0'(r)=\big(\Phi_0'\circ\Phi_0^{-1}\circ\varphi_\theta(r)\big)\circ y^{-1},\quad \varphi_\theta{}'\circ\varphi_\theta^{-1}\circ\Phi_0(r)=\varphi_\theta{}'(r)\circ y^{-1}.$$

Thus,

$$|\partial_r(f\circ y^{-1})|^2 = \Big\{(\partial_r f)\frac{\Phi_0'\circ\Phi_0^{-1}\circ\varphi_\theta(r)}{\varphi_\theta'(r)}\Big\}^2\circ y^{-1}.$$

Combining this with (4.13), (4.16) and (4.17), we obtain

$$\begin{aligned}
&|\nabla(f\circ y^{-1})|^2 = (\partial_r(f\circ y^{-1}))^2 + r^{-2}|\nabla_\theta(f\circ y^{-1})|^2\\
&\le \Big\{(\partial_r f)\frac{\Phi_0'\circ\Phi_0^{-1}\circ\varphi_\theta(r)}{\varphi_\theta'(r)}\Big\}^2\circ y^{-1}\\
&\quad + (\Phi_0^{-1}\circ\varphi_\theta(r))^{-2}\circ y^{-1}\Big\{(1+\varepsilon)(\partial_r f)^2\Big(\frac{|\nabla_\theta\varphi_\theta(r)|}{\varphi_\theta'(r)}\Big)^2 + (1+\varepsilon^{-1})|\nabla_\theta f|^2\Big\}\circ y^{-1}\\
&= (\partial_r f)^2\circ y^{-1}\Big\{\frac{(\Phi_0'\circ\Phi_0^{-1}\circ\varphi_\theta(r))^2}{(\varphi_\theta'(r))^2} + \frac{(1+\varepsilon)|\nabla_\theta\varphi_\theta(r)|^2}{(\varphi_\theta'(r))^2(\Phi_0^{-1}\circ\varphi_\theta(r))^2}\Big\}\circ y^{-1}\\
&+ \qquad (r\circ y^{-1})^{-2}|\nabla_\theta f|^2\circ y^{-1}\Big(\frac{(1+\varepsilon^{-1})r^2}{(\Phi_0^{-1}\circ\varphi_\theta(r))^2}\Big)\circ y^{-1}\\
&\le |\nabla f|^2\circ y^{-1}\max\Big\{\frac{(1+\varepsilon^{-1})r^2}{(\Phi_0^{-1}\circ\varphi_\theta(r))^2}, \frac{(\Phi_0'\circ\Phi_0^{-1}\circ\varphi_\theta(r))^2}{(\varphi_\theta'(r))^2} + \frac{(1+\varepsilon)|\nabla_\theta\varphi_\theta(r)|^2}{(\varphi_\theta'(r))^2(\Phi_0^{-1}\circ\varphi_\theta(r))^2}\Big\}\circ y^{-1}
\end{aligned}$$

for any $\varepsilon>0$. Therefore,

$$|\nabla(f\circ y^{-1})|^2 \le (\alpha|\nabla f|^2)\circ y^{-1} \tag{4.18}$$

and hence (4.2) follows from (4.14) by letting $g = f\circ y^{-1}$.

Finally, if h is constant then $\mu\circ y^{-1}$ is the standard Gaussian measure. Hence, by [2, Theorem 4.3] one has

$$\begin{aligned}
&W_2(\mu\circ y^{-1}, (f^2\circ y^{-1})\mu\circ y^{-1})^2 + (\mu\circ y^{-1})(f^2\circ y^{-1}\log f^2\circ y^{-1})\\
&\le 2\sqrt{2(\mu\circ y^{-1})(|\nabla(f\circ y^{-1})|^2)}\,W_2(\mu\circ y^{-1}, (f^2\circ y^{-1})\mu\circ y^{-1}).
\end{aligned}$$

By combining this with (4.18) we prove (4.5). □

Proof of Corollary 4.3. Since there exists a constant $c_0>0$ such that

$$\Phi_0'(r) = c_0 r^{d-1}\mathrm{e}^{-r^2/2} = \begin{cases}\Theta(r^{d-1}) & \text{as } r\to 0,\\ \Theta(r(1-\Phi_0(r))) & \text{as } r\to\infty,\end{cases}$$

where $f=\Theta(g)$ means that the two positive functions f and g are asymptotically bounded by each other up to constants, there exists a constant $c\ge 1$ such that

$$\frac{1}{c}\Phi_0'(r) \le \min\{r, r^{d-1}\}(1-\Phi_0(r)) \le c\Phi_0'(r), \quad r\ge 0.$$

Equivalently,

$$(4.19)\qquad \frac{1}{c}\Phi_0'\circ\Phi_0^{-1}(r)\le \min\{\Phi_0^{-1}(r),\ \Phi_0^{-1}(r)^{d-1}\}(1-r)\le c\Phi_0'\circ\Phi_0^{-1}(r),\quad r\in[0,1).$$

Next, it is easy to see from (4.6) that

$$(4.20)\qquad \Phi_0^{-1}\circ\varphi_\theta(r)=\begin{cases}\Theta(r^{\delta/2}) & \text{as } r\to\infty,\\ \Theta(r) & \text{as } r\to 0,\end{cases}$$

and

$$(4.21)\qquad \frac{1-\varphi_\theta(r)}{\varphi_\theta{}'(r)}=\frac{\int_r^\infty s^{d-1}\mathrm{e}^{V(s\theta)}\mathrm{d}s}{r^{d-1}\mathrm{e}^{V(r\theta)}}\le cr^{1-\delta}$$

for some constant $c>0$ and all $r\ge 1$. Combining (4.19), (4.20) and (4.21) we obtain

$$(4.22)\qquad \max\Big\{\frac{r^2}{(\Phi_0^{-1}\circ\varphi_\theta(r))^2},\ \frac{(\Phi_0'\circ\Phi_0^{-1}\circ\varphi_\theta(r))^2}{(\varphi_\theta{}'(r))^2}\Big\}\le c(1+r)^{2-\delta}$$

for some constant $c>0$.

If (4.7) holds then

$$|\nabla_\theta\varphi_\theta(r)|=|\nabla_\theta(1-\varphi_\theta(r)|\le c_4\min\Big\{r^d,\int_r^\infty s^{d-1}\mathrm{e}^{V(s\theta)}\mathrm{d}s\Big\},$$

so that due to (4.20) and (4.21)

$$\frac{|\nabla_\theta\varphi_\theta(r)|^2}{(\varphi_\theta{}'(r))^2(\Phi_0^{-1}\circ\varphi_\theta(r))^2}\le c_5\bigg(\frac{\min\{r^d,\int_r^\infty s^{d-1}\mathrm{e}^{V(s\theta)}\mathrm{d}s\}}{(r1_{\{r<1\}}+r^{\delta/2}1_{\{r\ge1\}})r^{d-1}\mathrm{e}^{V(r\theta)}}\bigg)^2\le c_6(1+r)^{2-3\delta}$$

for some constants $c_5,c_6>0$. Combining this with (4.22) and Theorem 4.2, we prove (4.8).

Finally, for any $x_1,x_2\in\mathbb{R}^d$ let $i\in\{1,2\}$ such that $|x_i|=|x_1|\vee|x_2|$. Similarly to the proof of Corollary 1.2, define

$$f(x)=\frac{|x-x_i|\wedge\frac{|x_i|}{2}}{(1+|x_i|)^{1-\delta/2}},\quad x\in\mathbb{R}^d.$$

Then

$$\Gamma(f,f):=(1+|\cdot|)^{2-\delta}|\nabla f|^2\le\frac{1_{\{|x_i|/2\le|\cdot|\le3|x_i|/2\}}(1+|\cdot|)^{2-\delta}}{(1+|x_i|)^{2-\delta}}\le C(\delta)$$

for some constant $C(\delta) > 0$. Since $|x_i| \geq \frac{1}{2}|x_1 - x_2|$, this implies that the intrinsic distance ρ induced by Γ satisfies

$$\rho(x_1, x_2)^2 \geq \frac{|f(x_1) - f(x_2)|^2}{C(\delta)} \geq C_1(\delta)\tilde{\rho}(x_1, x_2)^2$$

for some constant $C_1(\delta) > 0$, and hence is complete. Thus, by [25, Theorem 1.1] or [26, Theorem 6.3.3], (4.9) follows from (4.8). $\square$

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